CSCI567 Machine Learning (Spring 2021)

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Outline

1. Review of last lecture: Multi-armed Bandits

2. Reinforcement learning

Outline

Mulit-armed bandits: motivation

Imagine going to a casino to play a slot machine

- invariably it takes your money like a "bandit".

Of course there are many slot machines in the casino

- like a bandit with multiple arms (hence the name)
- **if I can play for 10 times, which machines should I play?**

Formal setup

There are K **arms** (actions/choices/...)

The problem proceeds in rounds between the environment and a learner: for each time $t = 1, \ldots, T$

- the environment decides the reward for each arm $r_{t,1}, \ldots, r_{t,K}$
- the learner picks an arm $a_t \in [K]$
- the learner observes the reward for arm a_t, i.e., r_{t,a_t}

Importantly, *learner does not observe rewards for arms not selected!*

This kind of limited feedback is now usually referred to as bandit feedback

Objective

Maximizing total rewards $\sum_{t=1}^{T} r_{t,a_t}$ seems natural

But the absolute value of rewards is not meaningful, instead we should compare it to some benchmark. A classic benchmark is

$$\max_{a \in [K]} \sum_{t=1}^{T} r_{t,a}$$

i.e. the largest reward one can achieve by always playing a fixed arm

So we want to minimize

$$\max_{a \in [K]} \sum_{t=1}^{T} r_{t,a} - \sum_{t=1}^{T} r_{t,a_t}$$

This is called the **regret**: *how much I regret for not sticking with the best fixed arm in hindsight?*

Balancing exploration vs. exploitation

A simple modification of "Greedy" leads to the well-known:

Upper Confidence Bound (UCB) algorithm

For $t = 1, \ldots, T$, pick $a_t = \operatorname{argmax}_a \ \mathsf{UCB}_{t,a}$ where

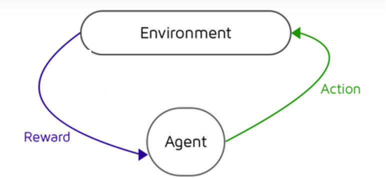
$$\mathsf{UCB}_{t,a} \triangleq \hat{\mu}_{t-1,a} + 2\sqrt{\frac{\ln t}{n_{t-1,a}}}$$

- the first term in $\mathsf{UCB}_{t,a}$ represents exploitation, while the second (bonus) term represents exploration
- the bonus term is large if the arm is not pulled often enough, which encourages exploration (adaptive due to the first term)
- a parameter-free algorithm, and *it enjoys optimal regret!*

Outline

Motivation

Multi-armed bandit is among the simplest decision making problems with limited feedback (**Bandit Feedback**).



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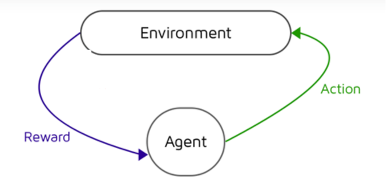

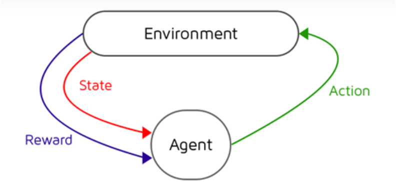

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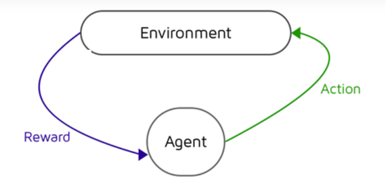

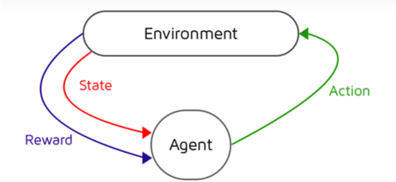

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- e.g. for Atari games, after making one move, the agent moves to a different state, with possible different rewards for each action

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The foundation of RL is **Markov Decision Process (MDP)**,
a combination of Markov model and multi-armed bandit

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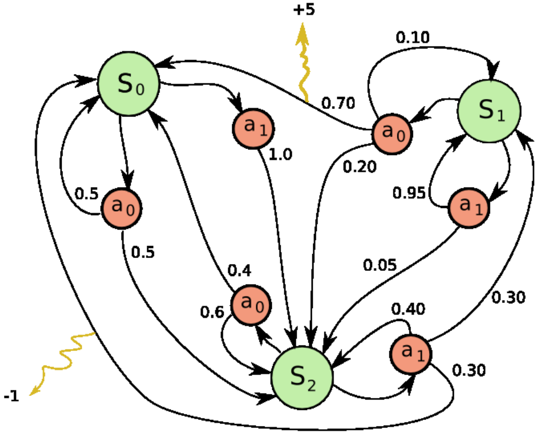
Different from Multi-armed bandit, the reward depends on the state.

Example

3 states, 2 actions



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Note: the discount factor allows us to consider an infinite learning process

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First goal: knowing all parameters, *how to find the optimal policy*

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V is called the **value function**. It satisfies the above **Bellman equation**: $|\mathcal{S}|$ unknowns, nonlinear, *how to solve it?*

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Knowing V, the optimal policy π^* is simply

$$\pi^*(s) = \operatorname*{argmax}_{a \in \mathcal{A}} \left(r_s(a) + \gamma \sum_{s' \in \mathcal{S}} P_a(s, s') V(s') \right)$$

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\end{aligned}$$

So the distance between V_k and V is shrinking *exponentially fast*.

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We discuss examples from two families of learning algorithms:

- **model-based** approaches
- **model-free** approaches

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Having estimates of the parameters we can then apply value iteration to find the optimal policy.

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- update the model parameters P, r
- update the value function V (via value iteration)

Model-free approaches

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Model-free approaches learn the Q function directly from samples.

Temporal difference

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α is like the learning rate

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for some learning rate α.

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There are many different algorithms and RL is an active research area.

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A brief introduction to some online decision making problems:

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  - learning the optimal policy with a known MDP: **value iteration**
  - learning the optimal policy with an unknown MDP: model-based approach and model-free approach (e.g. **Q-learning**)